Content

Tutorial 6
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BACK

Questions of today

1. Show that

$$\lim_{\epsilon\to 0^+} \frac{\zeta(1+\epsilon)+\zeta(1-\epsilon)}{2} = \gamma.$$

2. Evaluate $\zeta(-1)$.
3. Show that $\int_0^\infty \log(1-e^{-x}) = -\frac{\pi^2}{6}$
4. Show that

$$\sum_{n=1}^\infty \frac{\zeta(2n)}{2^{2n-1}} = 1.$$

Hints & solutions of today

1. Use Corollary 2.6 of lecture 11. (Or chapter 6 of textbook)
2. Use the formula appears in the page 3 of lecture 12 (or page 184 of text book):

$$\zeta(s) = \pi^{s-1/2}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}\zeta(1-s)$$

And the fact that $\zeta(2) = \frac{\pi^2}{6}$

3. Use the Taylor series expansion

$$\log(1-x) = -(x + \frac{x^2}{2} + \frac{x^3}{3})$$

4. One way is to use the formula

$$\zeta(2n) = \frac{1}{(2n-1)!}\int_0^\infty \frac{x^{2n-1}}{e^x-1}\,dx$$

So,

$$\sum_{n=1}^\infty \frac{\zeta(2n)}{2^{2n-1}} = \int_0^\infty \frac{\sinh(x/2)}{e^x-1}\,dx = \frac{1}{2}\int_0^\infty e^{-x/2}\,dx = 1.$$

Another way to do it is

$$\begin{aligned}\sum_{n=1}^\infty \frac{\zeta(2n)}{2^{2n-1}} &= \sum_{n=1}^\infty\sum_{k=1}^\infty \frac{2}{k^{2n}2^{2n}} \\ &= 2\sum_{k=1}^\infty\sum_{n=1}^\infty \frac{1}{(4k^2)^n} \\ &= 2\sum_{k=1}^\infty\sum_{n=1}^\infty \frac{\frac{1}{4k^2}}{1-\frac{1}{4k^2}} \\ &= 2\sum_{k=1}^\infty \frac{1}{4k^2-1} \\ &= \sum_{k=1}^\infty\left(\frac{1}{2k-1}-\frac{1}{2k+1}\right) \\ &= 1\end{aligned}$$
